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Topology and its Applications 66 (1995) 251–265

TOPOLOGY AND ITS APPLICATIONS

Complex surface singularities from the combinatorial point of view

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Received 2 September 1994; revised 17 February 1995

Abstract

We explore the interplay between graph theory and the topology of isolated singularities of complex surfaces. We study how certain geometric properties of surface singularities are reflected in the dual graphs of their resolutions. This provides new insights into the topology of surface singularities. For instance, we obtain a new characterization of the rational double points.

Keywords: Plumbing graph; Intersection matrix; Numerically Gorenstein; Spin; Adjunction formula; Canonical class

AMS classification: 32B99

0. Introduction

There exists a beautiful interplay between graph theory and the topology of isolated singularities of complex surfaces, which has been long studied by many authors, see for instance [5–7,13,14,4,19]: Given the germ of a normal surface singularity $(\mathcal{V}, P)$, we can take a good resolution of P, $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$. The set $E = \pi^{-1}(P)$ is a union of Riemann surfaces E_i intersecting normally. One constructs a graph Γ by taking a vertex for each E_i and joining each pair of different vertices by as many edges as there are points in the intersection of the corresponding E_i's. Each vertex has a genus g_i, that of E_i, and a weight w_i, the self-intersection number of E_i in $\widetilde{\mathcal{V}}$. We call the triple (Γ, w, g), with $g = (g_1, \ldots, g_n)$ and $w = (w_1, \ldots, w_n)$, the *dual graph* of $\widetilde{\mathcal{V}}$.

Conversely, consider a finite (undirected) (multi-)graph Γ with no loops. Let us endow Γ with a vector $w = (w_1, \ldots, w_n)$ of "weights", $w_i \in \mathbb{Z}$, and a vector of "genera"

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[1] Partially supported by CONACYT (Mexico), grant 1206-E92103, and by IMPA (Brasil).

SSDI 0166-8641(95)00038-0

$g = (g_1, \ldots, g_n)$, $g_i \in \mathbb{N}$. We call the triple (Γ, w, g) a *plumbing graph*. We may use the triple (Γ, w, g) to perform "plumbing", see for instance [5,14]. The result of plumbing is a compact smooth 4-manifold $X(\Gamma)$ with boundary; $X(\Gamma)$ is called a *graph manifold.* The topology of $X(\Gamma)$ is fully determined by the plumbing graph. If (Γ, w, g) is the dual graph of a resolution $\widetilde{\mathcal{V}}$, then the manifold $X(\Gamma)$ obtained by plumbing is diffeomorphic to $\widetilde{\mathcal{V}}$.

Given a plumbing graph one constructs, using the graph and the weights, a symmetric integral matrix $\mathbb{E}$ called the *intersection matrix.* By Grauert's Criterium [16, p. 43] together with [13] the plumbing graphs corresponding to resolutions of complex surface singularities are those whose intersection matrix $\mathbb{E}$ is negative definite.

Thus, it is interesting to understand the way how the geometric properties of surface singularities are reflected in the dual graphs of their resolutions. This work is a contribution in this direction.

In Section 1 we study which plumbing graphs have nonsingular intersection matrix and which ones have a definite intersection matrix. We show that given any finite graph Γ, the intersection matrix $\mathbb{E}$ is negative definite for almost every vector $w = (w_1, \ldots, w_n)$ of negative weights, so the corresponding graph manifold $X(\Gamma)$ is the resolution of some surface singularity.

Next we study, in Section 2, the dual graphs of surface singularities which are *numerically Gorenstein* in the sense of [3], i.e., singularities $(\mathcal{V}, P)$ such that $T\mathcal{V}^*$ is trivial over $\mathbb{C}$, where $\mathcal{V}^* = \mathcal{V} - \{P\}$. This is a subtle question: if we asked which surface singularities $(\mathcal{V}, P)$ have $T\mathcal{V}^*$ trivial over $\mathbb{R}$, the answer is easy: all of them. On the other hand, the Zarisky–Lipman conjecture (still open) claims that $T\mathcal{V}^*$ is never holomorphically trivial unless P is a regular point of $\mathcal{V}$.

By general arguments of topology, one can easily translate this question into a problem in linear algebra: given a plumbing graph (Γ, w, g) with nonsingular intersection matrix $\mathbb{E}$ of rank n, define its *canonical class* K to be the unique element in $\mathbb{Q}^n$ that satisfies the *Adjunction Formula*:

$$2g - \mathbf{2} = w + \mathbb{E}K,$$

where $\mathbf{2} = (2, \ldots, 2)$. Then, numerically Gorenstein means that the entries of K are integral numbers.

In Section 3 we push the question a little forward. We ask which (numerically Gorenstein) graph manifolds admit a spin structure (see [8] or Section 3 below for basics on spin manifolds); we call the corresponding triples (Γ, w, g) *spin graphs*. This is equivalent to demanding that K be even. We prove that given a weighted graph (Γ, w) with nonsingular intersection matrix $\mathbb{E}$, the existence of a vector of genera g for which (Γ, w, g) is spin is equivalent to the evenness of the weights in w. Of course one can push this question a little more: one can ask which graph manifolds actually admit a spin(3)-structure (see Section 3 below); this means $K = 0$. We do not know the answer to this question in general; however, if we also demand that $\mathbb{E}$ be negative definite, then the answer is given in Section 3 below (and it follows easily from [7]): The weights of (Γ, w, g) must be all -2, the genera are all 0 and Γ is one of the classical Dynkin diagrams A_n, D_n, E_6, E_7

or E_8. In fact this provides a characterization of the rational double points which, surprisingly, is not in [4]: These are the only singularities which have a resolution whose structure group reduces to $\mathrm{spin}(3) \cong \mathrm{SU}(2)$.

We also prove in Section 3 that if the intersection matrix $\mathbb{E}$ of (Γ, w, g) has odd determinant $\det(\mathbb{E})$, and if the weights are even, then the plumbing graph (Γ, w, g) is numerically Gorenstein if and only if it is spin. Thus, we are interested in knowing when the determinant of the intersection matrix $\mathbb{E}$ is odd. A beautiful answer comes from graph theory: first we observe that if the weights are even, this question is equivalent to demanding that the determinant of the matrix of adjacencies of Γ be odd; then we apply a formula due to F. Harary to show that this is equivalent to demanding that Γ has an odd number of spanning subgraphs of the form mK_2, the disjoint union of m copies of K_2, the graph with two vertices and one edge connecting them. For instance, the 1-skeletons of the tetrahedron, the cube and the icosahedron have odd determinant, but not those of the octahedron and the dodecahedron.

We may also look at this paper from a purely algebraic-combinatorial point of view. Start with a finite graph Γ with n vertices and no loops, and let $\mathbb{A}$ be the matrix of adjacencies of Γ. For each vector $w \in \mathbb{Z}^n$, define $\mathbb{E} = \mathbb{A} + \mathrm{diag}(w)$; we call w the *vector of weights*, and the pair (Γ, w) is a *weighted graph* with *intersection matrix* $\mathbb{E}$. We assume that w is such that $\mathbb{E}$ is nonsingular, which is the "generic" case (see Section 1 below). Call a vector $X \in \mathbb{Z}^n$ an *$\mathbb{E}$-characteristic vector* if it satisfies $\mathbb{E}X \equiv w \pmod 2$. Then, Section 2 below establishes a bijection between the space of all $\mathbb{E}$-characteristic vectors and the set of all vectors $g \in \mathbb{Z}^n$ such that the plumbing graph (Γ, w, g) is *n-Gorenstein*, i.e., its *canonical class* $K = \mathbb{E}^{-1}(2g - \mathbf{2} - w)$ is integral.

1. Surface singularities and plumbing graphs

Let $\mathcal{V}$ be (the germ of) a two dimensional complex analytic surface with a normal singularity at a point P, and let $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$ be a good resolution of P, where *good* means that the *exceptional divisor* $E = \pi^{-1}(P)$ has nonsingular irreducible components $E_1, \ldots, E_n$ which intersect normally. The topology of $\widetilde{\mathcal{V}}$ is fully determined by its associated *dual graph* $\Gamma = \Gamma(\widetilde{\mathcal{V}})$: the vertices of Γ correspond to the irreducible components of E, and two vertices v_i and v_j are joined by k edges whenever the corresponding curves E_i and E_j intersect at k points. Each vertex v_i of Γ has an associated *genus* g_i, the genus of the curve E_i, and a *weight* w_i, the Chern class of the normal bundle of E_i in $\widetilde{\mathcal{V}}$, which is equal to the self-intersection number E_i^2 of E_i in $\widetilde{\mathcal{V}}$.

Given the graph Γ, together with the weights $w = (w_1, \ldots, w_n)$ and genera $g = (g_1, \ldots, g_n)$, one can reconstruct $\widetilde{\mathcal{V}}$ up to diffeomorphism by performing plumbing (see for instance [14]). In particular, the topology of $\widetilde{\mathcal{V}}$ is completely determined by the triple (Γ, w, g).

Definition 1.1. A *plumbing graph* is a triple (Γ, w, g) consisting of a finite graph Γ with n vertices $v_1, \ldots, v_n$ and with no loops, a vector w of *weights*, $w = (w_1, \ldots, w_n)$,

$w_i \in \mathbb{Z}$, and a vector $g = (g_1, \ldots, g_n)$ of *genera*, $g_i \in \mathbb{N}$. Forgetting the genera, we get a *weighted graph* (Γ, w), whose *intersection matrix* $\mathbb{E} = \mathbb{E}(\Gamma, w)$ is defined as follows: let $\mathbb{A} = \mathrm{adj}(\Gamma)$ be the matrix of adjacencies of Γ, so $\mathbb{A} = (a_{ij})$ is a square symmetric $n \times n$ integral matrix with $a_{ij} = k$ if there are $k > 0$ edges of Γ between the vertices v_i and v_j and $a_{ij} = 0$ otherwise, so $\mathbb{A}$ has 0's in the diagonal; then $\mathbb{E} = \mathbb{A} + \mathrm{diag}(w)$: $\mathbb{E}$ equals $\mathbb{A}$ away from the diagonal, but its ith diagonal entry is w_i. Throughout this work we will assume that all our weighted graphs have nonsingular intersection matrix.

The following theorem tells us which plumbing graphs correspond to resolutions of complex surface singularities. This result is well known, it is an obvious corollary to [13] and Grauert's Contractibility Criterium (see [16, p. 43]).

Theorem 1.2. *A plumbing graph (Γ, w, g) is the dual graph of a good resolution $\widetilde{\mathcal{V}}$ of a normal surface singularity $(\mathcal{V}, P)$ iff its intersection matrix $\mathbb{E} = \mathbb{E}(\Gamma, w)$ is negative definite.*

If this is the case, $\mathbb{E}$ coincides with the matrix $\mathbb{E}(\widetilde{\mathcal{V}}) = (E_i \cdot E_j)$ of the intersection form of $\widetilde{\mathcal{V}}$ with respect to that basis of $H_2(\widetilde{\mathcal{V}}; \mathbb{Q})$ which consists of (the classes of) the irreducible components $E_1, \ldots, E_n$ of the exceptional divisor E of $\widetilde{\mathcal{V}}$.

Thus, given a graph Γ (with any genera for the vertices) one may ask whether there exist weights $w = (w_1, \ldots, w_n)$ such that the intersection matrix $\mathbb{E}$ is negative definite. We have the following result:

Theorem 1.3. *Given a fixed graph Γ, for all sufficiently large weights $w = (w_1, \ldots, w_n)$ the intersection matrix $\mathbb{E}$ is nonsingular. Moreover, for almost every $w \in \mathbb{Z}^n_+$, $\mathbb{E}$ is positive definite, and for almost every $w \in \mathbb{Z}^n_-$, $\mathbb{E}$ is negative definite.*

Therefore, for almost every set of negative weights for the vertices, the triple (Γ, w, g) is the dual graph of a good resolution of some surface singularity $(\mathcal{V}, P)$, for each vector of genera g. Here, by almost every we mean all but a finite number.

Proof. $\mathbb{E}$ is nonsingular if and only if its determinant $\det(\mathbb{E})$ is not zero; $\det(\mathbb{E})$ is a polynomial of the form:

$$\det(\mathbb{E}) = w_1 \cdots w_n + (\text{terms of lower degree in the } w_i\text{'s}),$$

hence $\det(\mathbb{E}) \neq 0$ whenever (the absolute values of) the weights are sufficiently large. Moreover, if $w \in \mathbb{Z}^n_+$ is sufficiently large, $\det(\mathbb{E})$ is positive, and if $w \in \mathbb{Z}^n_-$ is sufficiently large, $\det(\mathbb{E})$ is positive if n is even and negative for n odd.

Let $\chi(t) = \det(tI - \mathbb{E}) = t^n + c_1 t^{n-1} + \cdots + c_n$ be the characteristic polynomial of $\mathbb{E}$. Jacobi's criterium [12, p. 270] says that $\mathbb{E}$ is positive definite if and only if $c_i > 0$ for i even and $c_i < 0$ for i odd, while $\mathbb{E}$ is negative definite if and only if the c_i's are all positive. We also know that $(-1)^i c_i$ is the sum of all the principal $i \times i$ minors of $\mathbb{E}$.

If $\mathbb{E}$ is not positive definite, there exists an $i \in \{1, \ldots, n\}$ and a principal $i \times i$ minor m of $\mathbb{E}$ such that $m \leqslant 0$, hence there exists a subgraph Δ of Γ with i vertices and such that

$$m = \det\bigl(\mathrm{adj}(\Delta) + \mathrm{diag}(w|_\Delta)\bigr),$$

where $w|_\Delta$ consists of those weights that correspond to the vertices in Δ. But, the weights being positive, this can only happen for a finite number of $w|_\Delta$'s, hence the theorem.

In the negative definite case the arguments are similar: if $\mathbb{E}$ is not negative definite, there is an $i \in \{1, \ldots, n\}$ so that $c_i \leqslant 0$, and this can only happen for finitely many vectors with negative weights. □

2. On Gorenstein singularities

A normal surface singularity $(\mathcal{V}, P)$ is *Gorenstein* iff its dual sheaf is free at P, or equivalently if there exists a nowhere vanishing holomorphic 2-form on a punctured neighborhood of P in $\mathcal{V}$. For instance, every isolated complete intersection germ is Gorenstein. The following definition was introduced by Durfee in [3]:

Definition 2.1. The germ $(\mathcal{V}, P)$ is *numerically Gorenstein* iff the tangent bundle of $\mathcal{V} - \{P\}$ is topologically trivial over $\mathbb{C}$.

Gorenstein singularities are numerically Gorenstein, see for instance [3] or [17].

Definition 2.2. (i) Given a normal surface singularity germ $(\mathcal{V}, P)$ and a good resolution $\widetilde{\mathcal{V}}$ of P, the *canonical class* K of $\widetilde{\mathcal{V}}$ is the unique class in $H_2(\widetilde{\mathcal{V}}; \mathbb{Q}) \cong \mathbb{Q}^n$ that satisfies the *Adjunction Formula*

$$2g_i - 2 = E_i^2 + K \cdot E_i$$

for each irreducible component E_i of the exceptional divisor, see for instance [6,3].

In other words, the matrix $\mathbb{E} = \mathbb{E}(\widetilde{\mathcal{V}})$, being negative definite, determines a linear isomorphism

$$\mathbb{E} : \mathbb{Q}^n \cong H_2\bigl(X(\Gamma); \mathbb{Q}\bigr) \to H_2\bigl(X(\Gamma); \mathbb{Q}\bigr) \cong \mathbb{Q}^n$$

where n is the number of vertices in Γ, and the canonical class K is the inverse image under $\mathbb{E}$ of the vector

$$\boldsymbol{k} = \bigl(2g_1 - 2 - E_1^2, \ldots, 2g_n - 2 - E_n^2\bigr) \in \mathbb{Z}^n \subset \mathbb{Q}^n;$$

that is,

$$\mathbb{E}K = \bigl(2g_1 - 2 - E_1^2, \ldots, 2g_n - 2 - E_n^2\bigr).$$

(ii) Even if the plumbing graph (Γ, w, g) is not the dual graph of some $\widetilde{\mathcal{V}}$ as above, if the intersection matrix $\mathbb{E} = \mathbb{E}(\Gamma, w)$ is nonsingular we can still define the *canonical class* K of (Γ, w, g) as the unique vector $K \in \mathbb{Q}^n$ which satisfies the adjunction formula $\mathbb{E}K = 2g - \mathbf{2} - w$.

The following lemma is essentially proved in [3].

Lemma 2.3. *The germ $(\mathcal{V}, P)$ is numerically Gorenstein if and only if the canonical class K of some (equivalently: any) good resolution $\widetilde{\mathcal{V}}$ of P is in $\mathbb{Z}^n \subset \mathbb{Q}^n$, in other words, K is an integral linear combination of the E_i's.*

Proof. If $(\mathcal{V}, P)$ is numerically Gorenstein and $M = \partial\widetilde{\mathcal{V}}$, then $T\widetilde{\mathcal{V}}|_M$ is topologically trivial. Hence its first Chern class $c_1(\widetilde{\mathcal{V}})$ can be considered as a relative class in $H^2(\widetilde{\mathcal{V}}, M; \mathbb{Z})$. The dual of $c_1(\widetilde{\mathcal{V}})$ is $-K$, see [6,3,17], so K is integral. Conversely, if K is integral, then $c_1(\widetilde{\mathcal{V}})$ can be considered as a class in

$$H^2(\widetilde{\mathcal{V}}, M; \mathbb{Z}) \cong H_2(\widetilde{\mathcal{V}}; \mathbb{Z}).$$

Thus, by the naturality of the Chern classes, the 1st Chern class of $T\widetilde{\mathcal{V}}|_M$ vanishes: $c_1(T\widetilde{\mathcal{V}}|_M) = 0$; therefore the tangent bundle of $\widetilde{\mathcal{V}}$ is trivial over M, since the 2-dimensional complex bundles over a 3-manifold are classified by their 1st Chern class, hence the lemma. □

Definition 2.4. A plumbing graph (Γ, w, g) with nonsingular intersection matrix is *numerically Gorenstein* (or simply *n-Gorenstein*) iff its canonical class K is integral.

Lemma 2.5. *Let (Γ, w) be a weighted graph with nonsingular intersection matrix $\mathbb{E}$. Then there exists a vector g of genera such that the plumbing graph (Γ, w, g) is n-Gorenstein if, and only if, the congruence*

$$\mathbb{E}X \equiv w \pmod 2$$

has a solution in $\mathbb{Z}^n$.

Proof. If there exists one such vector g and if K is the corresponding canonical class, then $\mathbb{E}K = 2g - \mathbf{2} - w$, so $\mathbb{E}K \equiv w \pmod 2$. Conversely, given $X \in \mathbb{Z}^n$ such that $\mathbb{E}X \equiv w \pmod 2$, we define a vector

$$g = \frac{1}{2}(\mathbb{E}X + \mathbf{2} + w).$$

Then g is integral and X satisfies the adjunction formula

$$\mathbb{E}X = 2g - \mathbf{2} - w.$$

This proves the lemma except for one point: the vector g may have negative components, and a true vector of genera necessarily has (in this work) all components $\geqslant 0$. For this we claim that given the above vector X, we can always find an integral vector d such that $X' = X + 2d$ has a corresponding vector g' of genera which is integral and positive. Since $\mathbb{E}X' \equiv w \pmod 2$,

$$g' = \frac{1}{2}(\mathbb{E}X' + \mathbf{2} + w)$$

is integral, and since $\mathbb{E}X' = \mathbb{E}X + 2\mathbb{E}d$, $g' = g + \mathbb{E}d$. Thus the lemma will be proved if we show that given any constant $k > 0$, we can always find an integral vector d such that none of the components of $\mathbb{E}d$ is smaller than k. In fact, let $u = (k, \ldots, k)$, and let $v = \mathbb{E}^{-1}(u)$. Then v is in $\mathbb{Q}^n$. If v is integral, then we take $d = v$ and we have finished; otherwise we take $d = mv$, where m is any positive integer such that mv is integral. Then $\mathbb{E}d = m\mathbb{E}v = (mk, \ldots, mk)$, completing the proof. $\square$

Let $b : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ denote the integral bilinear form induced by $\mathbb{E}$, that is $b(X, Y) = X^{\mathrm{t}}\mathbb{E}Y$ for all $X, Y \in \mathbb{Z}^n$, and let $q(X) = b(X, X)$ be the corresponding quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$. We recall [11, p. 24] that a vector W is $\mathbb{E}$-*characteristic* (or b-*characteristic*) iff for every $X \in \mathbb{Z}^n$ we have

$$q(X) \equiv b(W, X) \pmod 2.$$

The following lemma shows that the vectors that satisfy the congruence in Lemma 2.5 are precisely the characteristic vectors of b.

Lemma 2.6. *A vector $W \in \mathbb{Z}^n$ is $\mathbb{E}$-characteristic if and only if $\mathbb{E}W \equiv w \pmod 2$, where $w = (w_1, \ldots, w_n)$ is the vector of weights.*

Proof. Let W be $\mathbb{E}$-characteristic, and let $e_1, \ldots, e_n$ be the usual basis of $\mathbb{Z}^n$; then $q(e_i) = w_i$ for all i, so the ith component of $\mathbb{E}W$ is $(\mathbb{E}W)^{\mathrm{t}}e_i = b(W, e_i) \equiv w_i \pmod 2$. Conversely, we note that for all vectors $x \in \mathbb{Z}^n$,

$$q(x) = \sum_i w_i x_i^2 + 2\sum_{i<j} a_{ij} x_i x_j \equiv w^{\mathrm{t}}x \pmod 2$$

and

$$b(W, x) = W^{\mathrm{t}}\mathbb{E}x = (\mathbb{E}W)^{\mathrm{t}}x \equiv w^{\mathrm{t}}x \pmod 2,$$

so we have $b(W, x) \equiv q(x) \pmod 2$. $\square$

We now let $\mathrm{Ch} = \mathrm{Ch}(\Gamma, w)$ be the set of $\mathbb{E}$-characteristic vectors:

$$\mathrm{Ch} = \{W \in \mathbb{Z}^n \mid W \text{ is } \mathbb{E}\text{-characteristic}\}.$$

The following lemma is essentially a consequence of [9]. The elegant proof of Proposition 2.8 below was done for us by Prof. Humberto Cárdenas.

Lemma 2.7. *The set* Ch *is infinite.*

Proof. Let C be the reduction modulo 2 of the intersection matrix $\mathbb{E}$ of (Γ, w). Ch is infinite if and only if the system

$$CX = d,$$

where d is the reduction of w modulo 2, has a solution over the two-element field $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Indeed, the liftings to $\mathbb{Z}^n$ of the solutions of this system are the $\mathbb{E}$-characteristic vectors by Lemma 2.6. In particular, we see that all characteristic vectors are congruent modulo 2 iff $CX = d$ has a unique solution, iff $\det(\mathbb{E})$ is odd. Since $\mathbb{E}$ is symmetric, so is C. Hence the lemma is a consequence of the following proposition.

Proposition 2.8. *Let $C \in \mathbb{Z}_2^{n\times n}$ be a symmetric matrix over $\mathbb{Z}_2$, and let $d \in \mathbb{Z}_2^n$ be the vector determined by the diagonal entries of C. Then there is always a solution $X \in \mathbb{Z}_2^n$ to the system*

$$CX = d.$$

Proof. If $n = 1$ or $d = 0$, the proposition is obviously true, so we may assume $n > 1$ and $d \neq 0$. We will make induction over n. We can always re-arrange the linear system so that $d_1 = c_{11} = 1$. For each $i > 1$ with $c_{i1} = 1$, we add to the ith equation the first one, so we get a new system $C'X = d'$ with $c'_{11} = 1$ and $c'_{i1} = 0$ for $i > 1$. Now for each $i > 1$ with $c'_{1i} = c_{1i} = 1$ we add to the ith column of C' the first one, thus obtaining a system $C''X = d'$ which admits a solution if and only if the original one does. C'' is of the form $(1) \oplus D$, where D is a symmetric $(n-1)\times(n-1)$ matrix whose diagonal entries are $d'_2, \ldots, d'_n$ and hence the induction hypothesis applies. □

The following is a corollary to Lemma 2.6 and the proof of Lemma 2.5.

Corollary 2.9. *Let (Γ, w) be a weighted graph with nonsingular intersection matrix $\mathbb{E}$.*

(i) *If the plumbing graph (Γ, w, g) is n-Gorenstein, then its canonical class $K \in \mathbb{Z}^n$ is an $\mathbb{E}$-characteristic vector.*

(ii) *Let* Ch *be as above and define $\mathcal{G} = \mathcal{G}(\Gamma, w)$ as the set*

$$\mathcal{G} = \{g \in \mathbb{Z}^n \mid (\Gamma, w, g) \text{ is n-Gorenstein}\}.$$

Then the map $\Psi : \mathrm{Ch} \to \mathcal{G}$ defined by $g = \Psi(W) = \frac{1}{2}(\mathbb{E}W + \mathbf{2} + w)$ is a bijection, with W being the canonical class of (Γ, w, g).

This last result tells us that the n-Gorenstein structures on a weighted graph are parameterized by their canonical classes. That is, if we fix the graph and the weight of each vertex, then there are as many choices of genera for the vertices with (Γ, w, g) Gorenstein, as vectors W in $\mathbb{Z}^n$ such that $\mathbb{E}W \equiv w \pmod 2$. Of course we are truly interested only in the vectors g which have nonnegative entries. We summarize the previous discussion in the following theorem:

Theorem 2.10. *Let (Γ, w) be an arbitrary weighted graph with nonsingular intersection matrix $\mathbb{E}$, and let b be its corresponding bilinear pairing. Let $\mathcal{G}$ be the set of all (integral) vectors of genera $g = (g_1, \ldots, g_n)$ for which (Γ, w, g) is n-Gorenstein. Then $\mathcal{G}$ is infinite and it is parameterized by the space* Ch *of b-characteristic vectors by means of the bijection $\Psi : \mathrm{Ch} \to \mathcal{G}$ given by*

$$\Psi(W) = \frac{1}{2}(\mathbb{E}W + \mathbf{2} + W),$$

where $\mathbf{2}$ is the n-vector $(2, \ldots, 2)$. The $\mathbb{E}$-characteristic vector W becomes the canonical class K for the corresponding numerically Gorenstein plumbing graph $(\Gamma, w, \Psi(W))$.

3. Spin structures

Let (Γ, w, g) be a plumbing graph and let $X(\Gamma)$ be the manifold obtained by plumbing [14]. So $X(\Gamma)$ is a compact 4-manifold with a complex structure in its interior and with boundary M. By taking some Hermitian metric, the structure group of the tangent bundle $TX(\Gamma)$ can be reduced to $U(2) \subset \mathrm{SO}(4)$. One has the bundle of complex orthonormal frames on $X(\Gamma)$, with fibre $U(2)$, contained as a subbundle of the bundle of frames with fibre SO(4). By definition, the group $\mathrm{spin}(n)$ is the nontrivial double cover of $\mathrm{SO}(n)$. There exists an inclusion $\mathrm{spin}(3) \to \mathrm{spin}(4)$, and also a projection $\mathrm{spin}(4) \to \mathrm{SO}(4)$. These induce corresponding maps between classifying spaces,

$$\mathrm{Bspin}(3) \to \mathrm{Bspin}(4) \to \mathrm{BSO}(4).$$

Definition 3.1 (cf. [8]). A *spin structure* on $X(\Gamma)$ means a lifting to Bspin(4) of the classifying map $\tau(\Gamma)$ of the tangent bundle $TX(\Gamma)$. A *spin*(3)*-structure* on $X(\Gamma)$ means a lifting of $\tau(\Gamma)$ further to Bspin(3). A plumbing graph (Γ, w, g) *admits* a spin (or a spin(3)) structure if the corresponding manifold $X(\Gamma)$ does. If (Γ, w, g) admits a spin structure, we say that it is a *spin graph*.

We remark that, properly speaking, a spin structure on $X(\Gamma)$ means a lifting of the classifying map $\tau(\Gamma)$ to some $\mathrm{Bspin}(n)$, n large, but this is equivalent to the definition above because $X(\Gamma)$ has nonempty boundary.

Proposition 3.2. *A plumbing graph (Γ, w, g) which is n-Gorenstein admits a spin structure iff its canonical class is even, i.e., $K \in 2\mathbb{Z}^n$, and it admits a spin*(3)*-structure iff $K = 0 \in \mathbb{Z}^n$.*

Proof. We know from [3] or [17] that $K \in H_2(X(\Gamma); \mathbb{Z})$ is the Poincaré–Lefschetz dual of $c_1(X(\Gamma), \mathcal{F})$, the 1st Chern class of $X(\Gamma)$ relative to some (any) trivialization $\mathcal{F}$ of its complex tangent bundle restricted to the boundary $\partial X(\Gamma)$. Hence K reduced modulo 2 is dual to the 2nd Stiefel–Whitney class $\omega_2(X(\Gamma), \mathcal{F})$ of $X(\Gamma)$ relative to $\mathcal{F}$. Thus, if K is even, then $\omega_2(X(\Gamma), \mathcal{F})$ vanishes. This implies that $\omega_2(X(\Gamma))$, the usual Stiefel–Whitney class of $X(\Gamma)$, also vanishes, because $\omega_2(X(\Gamma), \mathcal{F})$ maps to $\omega_2(X(\Gamma))$ under the standard homomorphism $j^* : H^2(X(\Gamma), \partial X(\Gamma); \mathbb{Z}) \to H^2(X(\Gamma); \mathbb{Z})$. Hence $X(\Gamma)$ is spin [8]. Conversely, if $X(\Gamma)$ is spin then $\omega_2(X(\Gamma)) = 0$, and since j^* is an isomorphism because $\mathbb{E}$ is nonsingular, $\omega_2(X(\Gamma), \mathcal{F})$ also vanishes. Hence K is even. The second statement, concerning spin(3)-structures, is proved similarly, just remembering that spin(3) is isomorphic to SU(2), and a complex surface admits a SU(2)-structure if and only if its first Chern class vanishes, see [17]. □

Theorem 3.3. *Let (Γ, w) be a weighted graph with nonsingular intersection matrix $\mathbb{E}$. If the weights w are even, then there exist infinitely many vectors g of genera, such that the triple (Γ, w, g) is spin. Conversely, if there is a vector g of genera such that the triple (Γ, w, g) is spin, then the weights are necessarily even.*

This contrasts with Theorem 3.10(iii) below, which says that if the intersection matrix $\mathbb{E}$ is negative definite and if there is a vector $g \in \mathbb{Z}_+^n$ such that the canonical class K is 0, then Γ is one of the classical Dynkin diagrams A_n, D_n, E_6, E_7 or E_8, the genera are all 0 and the weights are all -2.

Proof of Theorem 3.3. If $\mathbb{E}$ is nonsingular, we know from Section 2 above that given arbitrary weights, there exist vectors of genera for which the canonical class K is integral. First we claim that if the weights are even, then one can find a vector g of genera such that K is even. This is a consequence of the following result:

Lemma 3.4. *Let (Γ, w, g) be a n-Gorenstein graph with canonical class K, and suppose further that the weights are even. Let g' be the vector $g' = 2g - w/2 - 1$. Then (Γ, w, g') has canonical class $K' = 2K$, so this graph is spin.*

Proof.

$$\begin{aligned} K' &= \mathbb{E}^{-1}(2g' - \mathbf{2} - w) = \mathbb{E}^{-1}\left(2\left(2g - \frac{\omega}{2} - 1\right) - \mathbf{2} - w\right) \\ &= \mathbb{E}^{-1}(4g - \mathbf{4} - 2w) = 2K. \qquad \square \end{aligned}$$

Let us return to the proof of Theorem 3.3. If the vector g' above has components $\geqslant 0$, then Lemma 3.4 implies that (Γ, w, g') is spin. Otherwise, just as in the proof of Lemma 2.5 above, we choose a positive vector u and we let d be a vector such that $\mathbb{E}d = mu$ for some positive m. We let K' and g' be as in Lemma 3.4, and we set $X^* = K' + 2d$, so that X^* is also $\mathbb{E}$-characteristic and it is even. The vector of genera corresponding to X^* is

$$g^* = \frac{1}{2}\left(w + \mathbb{E}X^* + \mathbf{2}\right) = g' + mu.$$

Hence, we can assume that g^* is positive by choosing u large enough. Thus, given g such that (Γ, w, g) is n-Gorenstein, for every sufficiently large positive vector d we get a vector g^* as above, such that (Γ, w, g) is spin. This proves the first claim in the theorem. Let us now prove the converse that if (Γ, w, g) is spin for some g, then the weights w are necessarily even. This is easy: we know that

$$w = 2g - \mathbf{2} - \mathbb{E}K,$$

and $\mathbb{E}K$ is even because K is even, hence w is even. $\square$

Remarks 3.5. Let (Γ, w, g) be a spin graph, $\mathbb{E}$ its intersection matrix, and b, q, the corresponding integral bilinear and quadratic forms.

(i) Since w is even, $q(X) \in 2\mathbb{Z}$ for all $X \in \mathbb{Z}^n$, i.e., q is an *even* (or *Type II*) quadratic form [11].

(ii) Since the canonical class K is also even, $K^2 = q(K) \equiv 0 \pmod 8$.

(iii) Let us denote by $\sigma = \sigma(\mathbb{E})$ the *signature* of $\mathbb{E}$ (or of b): counted with their multiplicities, σ is the number of positive eigenvalues of $\mathbb{E}$ minus the number of negative ones. Also, denote by $s = \det(\mathbb{E})/|\det(\mathbb{E})|$ the *sign* of $\det(\mathbb{E})$. If $\det(\mathbb{E})$ is odd, by the theorem in [9] we have that

$$q(K) - \sigma \equiv \det(\mathbb{E}) - s \pmod 4,$$

and using (ii) we get

$$\sigma \equiv s - \det(\mathbb{E}) \pmod 4,$$

so in particular $\det(\mathbb{E})$ odd implies σ even.

Lemma 3.6. *If the weights of Γ are even and* $\det(\mathbb{E})$ *is odd, then (Γ, w, g) is n-Gorenstein if and only if it is spin.*

Proof. Let C be the reduction of $\mathbb{E}$ modulo 2. Then C determines a homomorphism

$$C : \mathbb{Z}_2^n \to \mathbb{Z}_2^n.$$

This is an isomorphism when the determinant of $\mathbb{E}$ is odd. If (Γ, w, g) is n-Gorenstein, the canonical class K is integral, so the canonical class reduced modulo 2, $K|_2$, lives in $\mathbb{Z}_2^n$; by definition K satisfies the adjunction formula $\mathbb{E}K = 2g - \mathbf{2} - w$, so that $C(K|_2) = -w|_2 = 0 \in \mathbb{Z}_2^n$, because w is even. Therefore K is even, because C is an isomorphism. □

Thus we see that for plumbing graphs with even weights and odd determinant, n-Gorenstein structures are automatically spin. This statement is definitely false for plumbing graphs whose intersection matrix has even determinant, as it will be illustrated shortly.

Examples 3.7. First consider a plumbing graph with only one vertex:

$$\overset{\bullet}{(w, g)}$$

In this case the weight w equals the determinant of the intersection matrix $\mathbb{E} = (w)$, hence the determinant is odd when w is odd. Assuming $w \neq 0$, the adjunction formula now reads:

$$K = \frac{2g-2}{w} - 1.$$

Hence (Γ, w, g) is n-Gorenstein iff $2g - 2 \equiv 0 \pmod w$. The graph is spin iff

$$2g - 2 \equiv w \pmod{2w}.$$

For instance, if $w = -1$, then $K = 1 - 2g$, so the graph is always n-Gorenstein, but it is never spin. If $w = -2$, $K = -g$ and the graph is always n-Gorenstein, but it is spin only when g is even.

Consider now the complete graph $\Gamma = K_2$ which two vertices:

$$\bullet \!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet$$

Let $w = (w_1, w_2) \in 2\mathbb{Z}^2$ be any pair of even weights. The intersection matrix is

$$\mathbb{E} = \begin{pmatrix} w_1 & 1 \\ 1 & w_2 \end{pmatrix}.$$

The determinant is $d = w_1 w_2 - 1$, so it is odd. By Section 2 above, there exist an infinite number of vectors of genera $g = (g_1, g_2) \in \mathbb{N}^2$ such that the plumbing graph (Γ, w, g) is n-Gorenstein; let g be one such, and let $K = (m_1, m_2)$ (or $K = m_1 E_1 + m_2 E_2$ in geometric language) be the canonical class. The adjunction formula implies:

$$\begin{aligned} m_1 &= 2g_2 - 2 - w_2(1 + m_2), \text{ and} \\ m_2 &= 2g_1 - 2 - w_1(1 + m_1); \end{aligned}$$

so K is automatically even and the graph is spin.

If m is a positive integer, then mK_2 denotes the disjoint union of m copies of the complete graph K_2 with two vertices. A subgraph Γ' of Γ is a *spanning subgraph* if Γ' has all the vertices of Γ. A spanning subgraph of Γ is called a *1-factor* of Γ if it is of the form mK_2. Note that the existence of a 1-factor for Γ implies that n is even.

Theorem 3.8. *Let (Γ, w) be a graph with even weights and nonsingular intersection matrix $\mathbb{E}$. The following statements are equivalent:*

(i) *(Γ, w, g) is n-Gorenstein if and only if (Γ, w, g) is spin, for every $g \in \mathbb{Z}^n$.*
(ii) *The determinant of $\mathbb{E}$ is odd.*
(iii) *The space Ch of $\mathbb{E}$-characteristic vectors is $(2\mathbb{Z})^n$.*
(iv) *The determinant of the matrix $\mathbb{A}$ of adjacencies of Γ, is odd.*
(v) *The number of 1-factors of Γ is odd (and so n is even).*

Proof. The last claim in Theorem 2.10 and the first one in Proposition 3.2, together with (i), imply that all $\mathbb{E}$-characteristic vectors are 0 modulo 2, and this implies (ii) by a remark in the proof of Lemma 2.7. By Lemma 3.6, (i) follows from (ii). That (i) and (iii) are equivalent follows from Theorem 2.10 using that

$$\Psi(2\mathbb{Z}^n) = \{g \in \mathbb{Z} \mid (\Gamma, w, g) \text{ is spin}\}.$$

The equivalence of (ii) and (iv) follows from the fact that $\mathbb{A}$ and $\mathbb{E}$ are congruent modulo 2 because the weights are even. That (iv) and (v) are equivalent is shown in [2, p. 86] but for the reader's convenience we repeat the argument here. The proof is based in Harary's (or Coates') formula [1, p. 40; 2, p. 32]:

$$\det(\mathbb{A}) = \sum_{\Delta} (-1)^{r(\Delta)} 2^{s(\Delta)},$$

where the sum runs over all spanning subgraphs Δ of Γ whose connected components are either circuits or copies of K_2, $r(\Delta)$ is n minus the number of connected components of Δ, and $s(\Delta)$ is the number of components of Δ which are circuits. For each Δ in the sum above, the corresponding term is odd if and only if $s(\Delta) = 0$, which means that

Δ is a 1-factor. Hence, det($\mathbb{A}$) can only be odd if Δ has a 1-factor and so n is even. Since all 1-factors of Δ appear in Harary's sum, and all contribute to the sum with the same term $(-1)^{n/2}$, we see that det($\mathbb{A}$) is odd if and only if Γ has an odd number of 1-factors. □

Examples 3.9. For the following graphs Γ, det($\mathbb{A}$) is odd (see [1, p. 17, 39, 145, 152]).

(i) The 1-skeletons of the tetrahedron, the cube and the icosahedron, but not those of the octahedron and the dodecahedron.

(ii) The complete graph K_n with n vertices, if n is even.

(iii) The 1-skeleton Q_k of the standard cube $[0,1]^k \subset \mathbb{R}^k$, for k odd.

(iv) The path A_n with n vertices and length $n-1$, for n even.

Let us now restrict our attention to plumbing graphs that arise from complex singularities. We have the following theorem.

Theorem 3.10. *Let (Γ, w) be a weighted graph with $n \geqslant 1$ vertices and with negative definite intersection matrix $\mathbb{E}$. Then:*

(i) *There are infinitely many different genera $g = (g_1, \ldots, g_n)$ for which (Γ, w, g) is the dual graph of a resolution of some surface singularity $(\mathcal{V}, P)$ which is numerically Gorenstein (i.e., K is integral).*

(ii) *If the weights $w = (w_1, \ldots, w_n)$ are even, then there are infinitely many different genera $g = (g_1, \ldots, g_n)$ for which the corresponding singularity is spin (i.e., K is even). If some weight is odd, K can not be even (for any g).*

(iii) *If $K = 0$ for some g, then the weights are all -2, the genera are all 0 and Γ is one of the classical Dynkin diagrams A_n, D_n, E_6, E_7 or E_8. The corresponding singularity is a rational double point.*

(iv) *If the weights are even and the determinant of $\mathbb{E}$ is odd, then for every $g \in \mathbb{N}^n$ we have that (Γ, w, g) is numerically Gorenstein if and only if (Γ, w, g) is spin. (Recall that every isolated complete intersection germ is automatically Gorenstein, hence it is numerically Gorenstein.)*

(v) *If a resolution $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$ of some n-Gorenstein surface singularity $(\mathcal{V}, P)$ is spin, then it is minimal. That is, if $\pi' : \widetilde{\mathcal{V}}' \to \mathcal{V}$ is a resolution of P, then there exists a holomorphic map $h : \widetilde{\mathcal{V}}' \to \widetilde{\mathcal{V}}$ such that $\pi' = \pi \circ h$.*

Proof. Statements (i), (ii) and (iv) are already proved. Statement (v) follows from (ii) together with Castelnuovo's Criterium for minimality (see [16, p. 42]). For (iii) we note that $K = 0$ implies $2g_i - 2 = w_i$ for all i, by the adjunction formula; since $w_i < 0$ and $g_i \geqslant 0$ this implies $w_i = -2$ and $g_i = 0$ for all i. The rest follows from a theorem of Hirzebruch (see [4, p. 136]) saying that the above mentioned Dynkin diagrams are the only graphs weighted by -2's, whose intersection matrix is negative definite. □

Examples 3.11. Let us consider some hypersurface germs in $\mathbb{C}^3$. Hypersurface germs are automatically Gorenstein, so their canonical class is integral.

(a) Consider the homogeneous singularity

$$\{X^k + Y^k + Z^k = 0\}, \quad k > 1.$$

We know from [10] or [15] that the minimal resolution is a holomorphic line bundle with Chern class $-k$ over a Riemann surface S of genus

$$g = 1 + \frac{k(k-3)}{2}.$$

Its canonical class is $K = (2-k)S$, hence it is spin iff k is even.

(b) The surface singularity

$$X^p + Y^{pq} + Z^{pq} = 0, \quad p, q \geqslant 2.$$

By [10] its minimal resolution is a holomorphic line bundle with Chern class $-p$ over a Riemann surface S of genus

$$g = 1 + \frac{p(pq-q-2)}{2}.$$

An easy computation, using the adjunction formula, shows that

$$K = (q - pq + 1)S.$$

Hence, this singularity is spin iff p is even and q is odd.

(c) Consider the singularity

$$\{X^p + Y^q + Z^{pq} = 0\},$$

with $p, q \geqslant 2$ relative primes. Again the minimal resolution $\widetilde{\mathcal{V}}$ is a line bundle over a compact Riemann surface S. The Chern class of $\widetilde{\mathcal{V}}$ is -1, and the genus of S is

$$g = \frac{pq - p - q + 1}{2}.$$

The canonical class is $K = (p + q - pq)S$, and $p + q - pq$ is always odd, because p and q can not be both even. Thus this singularity is never spin.

References

[1] N. Biggs, Algebraic Graph Theory, Cambridge Tracts in Mathematics 67 (Cambridge University Press, Cambridge, 1974).

[2] D.M. Cvetković, M. Doob and H. Sachs, Spectra of Graphs, Pure and Applied Mathematics 87 (Academic Press, Boston, MA, 1979).

[3] A. Durfee, The signature of smoothings of complex surface singularities, Math. Ann. 232 (1978) 85–98.

[4] A. Durfee, Fifteen characterizations of rational double points and simple singularities, Enseign. Math. 25 (1979) 131–163.

[5] F. Hirzebruch et al., Differential Manifolds and Quadratic Forms (Marcel Dekker, New York, 1972).

[6] F. Hirzebruch, Hilbert modular surfaces, Enseign. Math. 4 (1973) 183–281.

[7] F. Hirzebruch, Über Singularitäten komplexer Flächen, Rend. Math. 25 (1968) 213–232.

[8] H.B. Lawson and M.L. Michelsohn, Spin Geometry (Princeton University Press, Princeton, NJ, 1989).

[9] W. Ledermann, An arithmetical property of quadratic forms, Comment. Math. Helv. 33 (1959) 34–37.

[10] J. Milnor, On the 3-dimensional Brieskorn manifolds $M(p,q,r)$, in: L. Neuwirth, ed., Knots, Links and 3-Manifolds, Annals of Mathematics Studies 84 (Princeton University Press, Princeton, NJ, 1975) 175–225.

[11] J. Milnor and D. Husemoller, Symmetric Bilinear Forms, Ergebnisse der Mathematik und Ihrer Grenzgebiete 73 (Springer, Berlin, 1973).

[12] A.I. Maltsev, Fundamentos de Algebra Lineal (Mir, Moscú, 1976).

[13] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. IHES 9 (1961) 229–246.

[14] W.D. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Trans. Amer. Math. Soc. 268 (1981) 299–344.

[15] P. Orlik and Ph. Wagreich, Isolated singularities of algebraic surfaces with $\mathbb{C}^*$-action, Ann. of Math. 93 (1971) 205–228.

[16] P. Orlik, Seifert Fiber Spaces, Lecture Notes in Mathematics 291 (Springer, Berlin, 1972).

[17] J.A. Seade, Singular points of complex surfaces and homotopy, Topology 21 (1982) 1–8.

[18] J.A. Seade and B.F. Steer, Complex singularities and the framed cobordism class of compact quotients of 3-dimensional Lie groups by discrete subgroups, Comment. Math. Helv. 65 (1990) 349–374.

[19] S.S.-T. Yau, The multiplicity of isolated two-dimensional hypersurface singularities, Amer. J. Math. 111 (1989) 753–767.